Lecture 1 Lessons learned

- Topology concerns properties invariant under deformations
  e.g. simple solutions from deformed objects

- Topology ubiquitous in (theoretical) physics
  ↳ course on condensed matter, topological phases, elemental aspects of anyons, adiabatic quantum phenomena

- Basic elements of topology

Lecture 2

Continuity & metric spaces

Key aspect:
- Reconcilation of set theoretic topology with metric understanding of continuity (no jumps)
- Construction of new topological spaces

interlude: **Short repetition about notation**

$A \subseteq B$ — A is a subset of B

$\{\}$, $\emptyset$ — empty set (set with zero elements)

2^A — family of all subsets of A
e.g. $A = \{1,2\}$, then $2^A = \{\{\}, \{1\}, \{2\}, \{1,2\}\}$.

$A \cup B$ union: all elements from A & B.
$= \{x \mid x \in A \text{ or } x \in B\}$

$A \cap B$ intersection: all common elements of A & B
$= \{x \mid x \in A \ \& \ x \in B\}$.

Suggestions

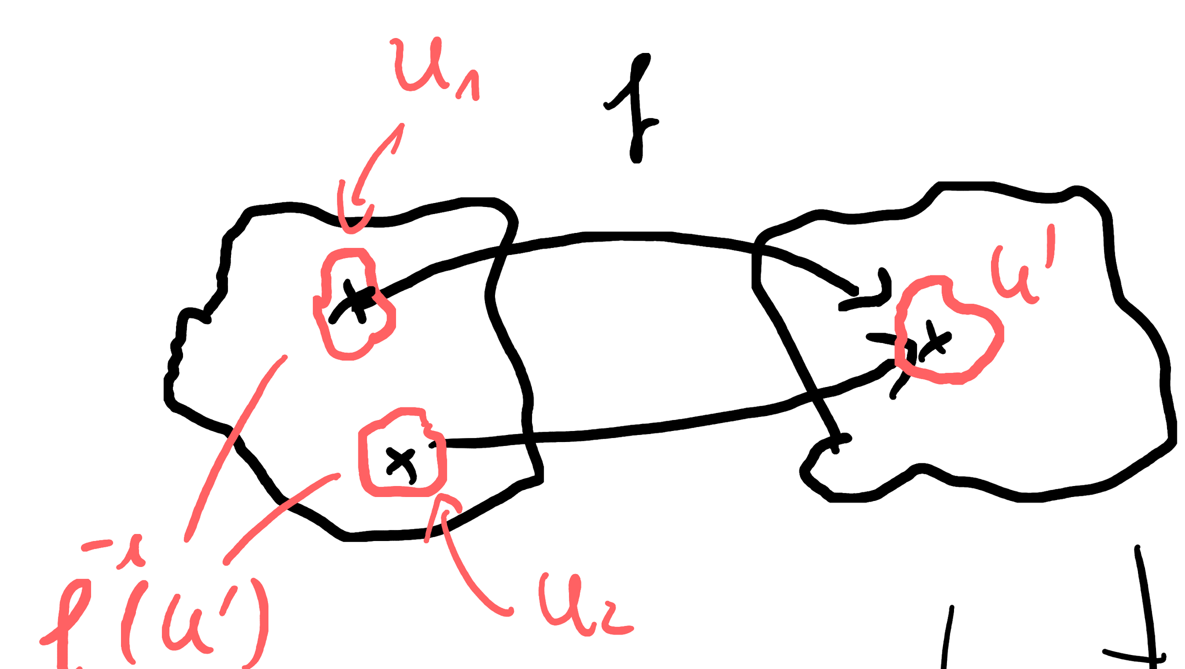
① Create own vocabulary list & send them to us, we include them into the script

② You can create cheat sheet for exam. Official cheat sheet will be distributed for use. Send your suggestions (latex!)

I.2. Continuity

preimage & inverse map: For sets X, X', a map $f: X \to X'$, and a subset $U' \subseteq X'$, the preimage U of U' under are all elements of X that are mapped into U'.

$$U = \{x \in X \mid f(x) \in U'\}$$



The function $f^{-1}: 2^{X'} \to 2^{X}$ that maps a subset $U' \subseteq X'$ to its preimage under f is the <u>inverse map</u> of f. We have

$$f(f^{-1}(U')) = U' \quad \& \quad f^{-1}(f(U)) \supseteq U$$

where $f(X) = \{f(x) \mid x \in X\}$ extends the function f to subsets of X.

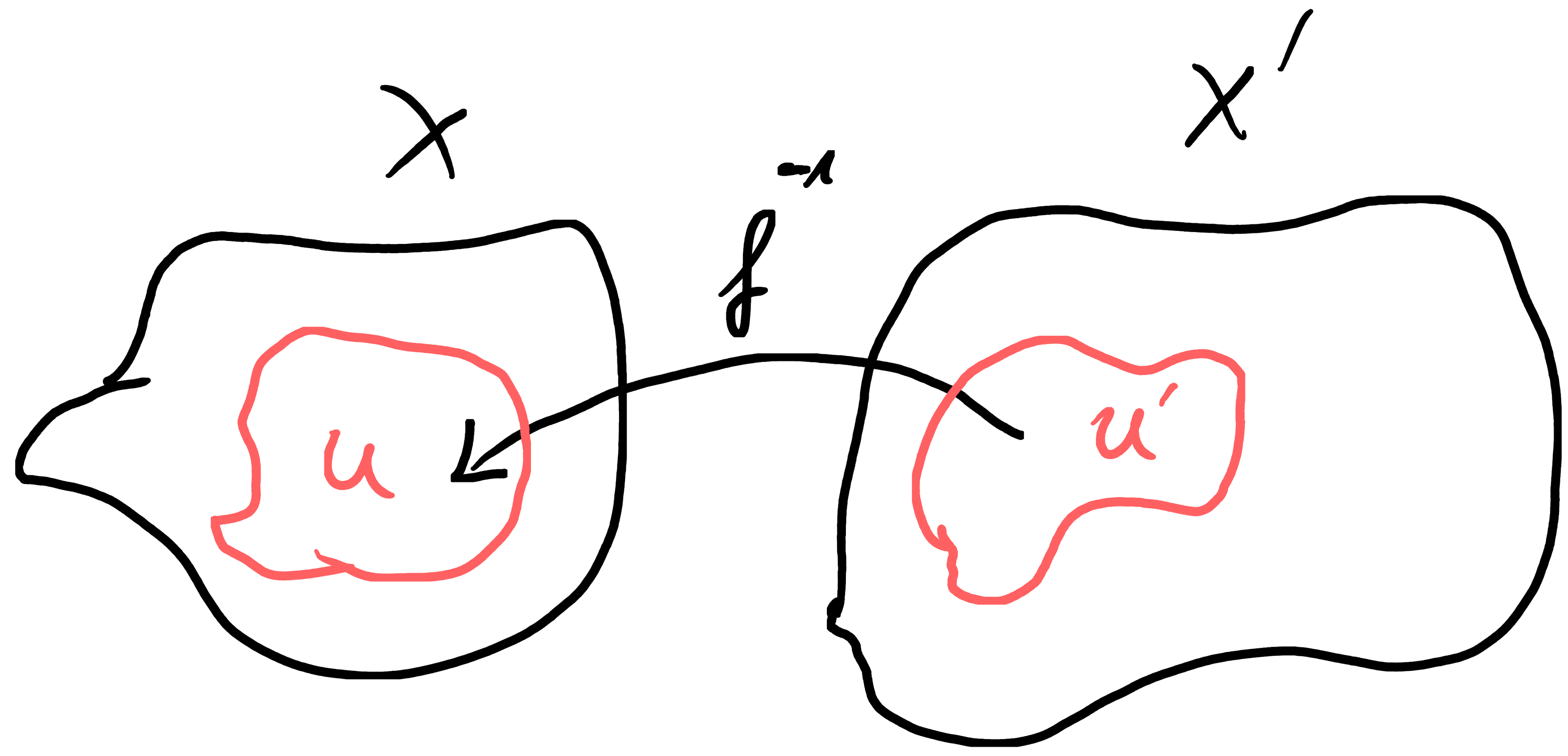
2^X: Family of all subsets of X

→ e.g. $f^{-1}(f(U_1)) = U_1 \cup U_2 \supseteq U_1$

continuous function

For topological spaces (X,τ) & (X',τ'), a function that $f: X \to X'$ is continuous if $f^{-1}(U')$ is open for every open U'.

The function f is continuous at $x \in X$ if there is a neighborhood U of x such that f constrained to that neighborhood is continuous, i.e., $\tilde{f}: U \to X'$, $x \mapsto f(x)$ is continuous



U' open $\Rightarrow$ U open

U open $\Rightarrow$ U' open is not a good definition, see exercises

equivalence of topological spaces

homeomorphism A bijection $f: X \to X'$ between topological spaces X & X' is a homeomorphism if f & f^{-1} are continuous. X & X' are homeomorphic if there is such a bijection between them

I.3. Metric spaces

Comment:
Are the set theoretic definitions of open & continuous the same as the ones familiar to us?

A metric space is a space with a "distance", which induces a topology (as we will see).

Let X be a set & $d: X \otimes X \to \mathbb{R}$ a "distance" function with, for $x, y \in X$

1) $d(x,x) = 0$ — no distance from point to itself

2) $d(x,y) > 0$ if $x \neq y$ — positive distance

3) $d(x,y) = d(y,x)$ — symmetry

4) $d(x,z) \geq d(x,y) + d(y,z)$ — triangle inequality (no shortcuts rule)

Axioms partially redundant, 1 & 2 can be combined to

$$d(x,y) = 0 \iff x = y$$ identity of indiscernibles

Exercise about that

The pair (X, d) is called metric space

<u>open ball</u> An open ball around $x \in X$ with radius $r \geq 0$

$$B_x(r) = \{ y \in X \mid d(x,y) < r \}.$$

$$B_x(0) = 0.$$

open (definition from metric spaces)

A subset $\mathcal{U}$ of a metric space is open if for every $x \in \mathcal{U}$ there is a distance $r_x > 0$ such that $B_x(r_x)$ is in $\mathcal{U}$.

metric-induced topology

Let (X, d) be a metric space, then the metric-induced topology τ is the collection of the above-mentioned sets

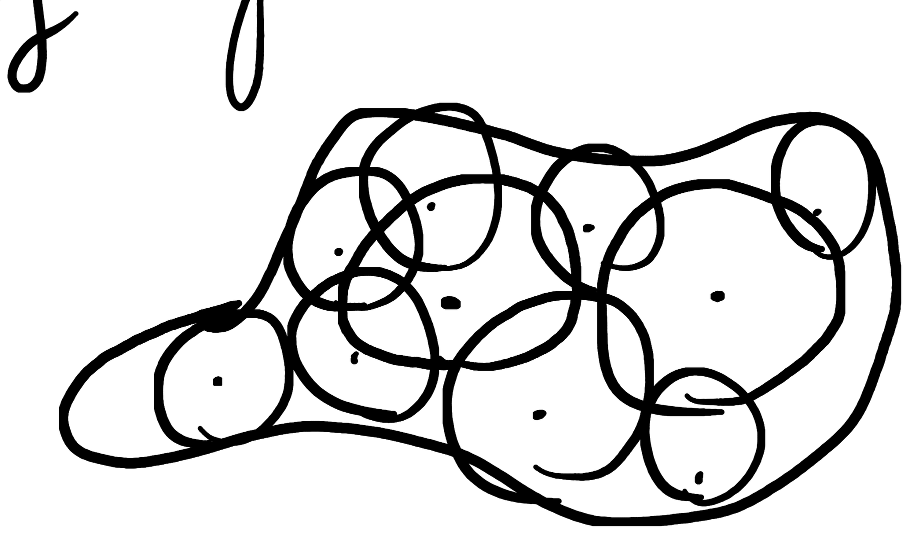
Alternatively, the metric-induced topology is constructed by arbitrary unions of open balls

$\mathcal{U} \in \tau$ if & only if $\mathcal{U} = \bigcup_{i \in I} B_{x_i}(r_i)$ with an arbitrary index set I.

I.e., open balls form a base for the metric-induced topology.

proof that both definitions of open are the same → exercises.

Essential for equivalence is that every open set is the union of open balls



→ exercises