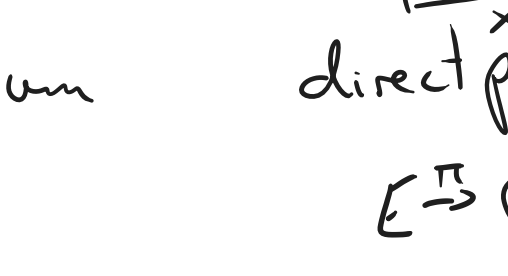
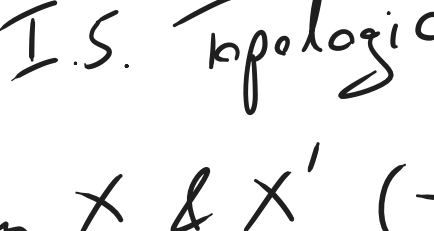


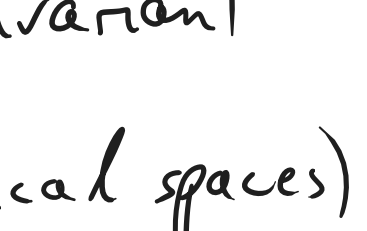
Lecture IV Topological invariants



last lecture Const. of top. spaces



direct sum



direct product

$$E \cong B, (E, \pi, B, F)$$

$$S^1 \cong \mathbb{R}/\mathbb{Z}$$

quotient space

base space

fiber

I.S. Topological invariant

Given X & X' (topological spaces), deciding whether $X \cong X'$ is hard (generally). Ex. $SO_3 \cong GL(3, \mathbb{R})$

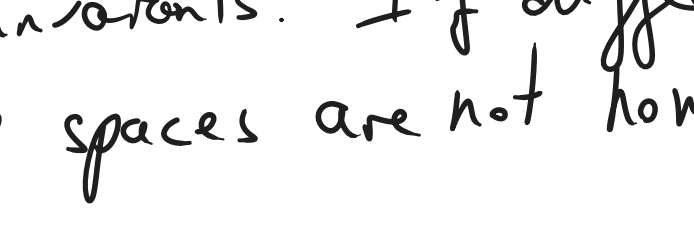
homeomorphic to S^3 ?

topological invariants (colloquial) Any property of top. space conserved by cont. functions.

A topological invariant is a function $f: \text{Top} \rightarrow \mathcal{C}$, where Top is the category of top. spaces & $\mathcal{C}$ is an arbitrary set, with the property

$$f(X) = f(i(X)), \text{ where } i: X \rightarrow Y \text{ is an homeomorphism. } [X, Y \in \text{Top.}]$$

Examples

- Cardinality $|X|$ of the space
- Cardinality of topology $|\tau|$
- separation axioms/properties
- metrizability
- numbers of holes 
- homology groups, cohomology groups
- Chern number, Pontryagin numbers

Idea: Given two top. spaces, calculate (a number of) topological invariants. If different values are assumed $\Rightarrow$ spaces are not homeomorphic

$$X \xrightarrow{f} X'$$

$$C_1 \xrightarrow{f} C_2$$

I.S.1 Homotopy groups
puts intuition of "continuously deformation" on firm grounds.

homotopy/homotopic

Two continuous maps $f, g: X \rightarrow X'$ are homotopic if there is continuous function $\mathcal{H}: X \times [0, 1] \rightarrow X'$ with the properties

$$\mathcal{H}(x, 0) = f(x) \text{ and } \mathcal{H}(x, 1) = g(x).$$

Example: $X = [0, 1], X' = \mathbb{R}^2$ $\mathcal{H}(x, 0) = f$ $\mathcal{H}(x, 1) = g$ $\mathcal{H}$ is called a homotopy.

We write $f \sim g$.

point contractible

A continuous map $f: X \rightarrow X'$ is point contractible, if it is homotopic to a constant function.



Def. Homotopy classes

Given top. spaces X, X' & functions $A: \{f: X \rightarrow X' \mid f \text{ cont.}\}$

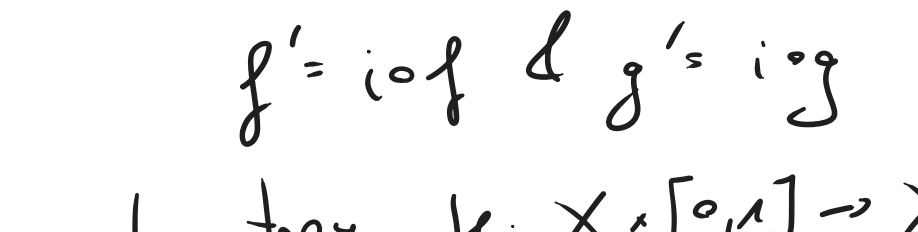
The $f \sim g$ defines an equivalence relation [see ex.]

& the equivalence classes are homotopy classes.

1) point contract. $\circ$ $\circ$

2) encircle $(0,0)$ n times $\circ$ $(k=2)$.

Idea: Use homotopy classes of loops to find "1D holes"



I.S.1.1. Homotopies as topological invariant.

For top. spaces X, Y and homotopic functions $f \sim g, f, g: X \rightarrow Y$, and a homeomorphism $i: Y \rightarrow Y'$, then

$f' = i \circ f$ & $g' = i \circ g$ are homotopic.

Proof: Take a homotopy $\mathcal{H}: X \times [0, 1] \rightarrow Y$ with

$$\mathcal{H}(x, 0) = f \text{ and } \mathcal{H}(x, 1) = g, \text{ then}$$

$$\mathcal{H}'(x, t) = i(\mathcal{H}(x, t)) \text{ with } (x, t) \mapsto i(\mathcal{H}(x, t)) \text{ is continuous \& } \mathcal{H}'(x, 0) = i(\mathcal{H}(x, 0)) = i(f(x)) = (i \circ f)(x) = f'(x)$$

$$\mathcal{H}'(x, 1) = i(\mathcal{H}(x, 1)) = \dots = g'(x).$$

$\Rightarrow \mathcal{H}'$ is a homotopy & $f' \sim g'$.

comment
concatenation of continuous functions
 $\Rightarrow$ continuous, see script

the fundamental group π_1

Consider a topological space X & a sp. $x_0 \in X$.

$$\mathcal{L} = \{f: S^1 \rightarrow X \mid f(0) = f(1) = x_0, \text{ cont.}\} \quad S^1 = [0, 1] / (0 \sim 1)$$

Concatenation of paths $f, g: S^1 \rightarrow X$ is $(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \in \mathcal{L}$

Concatenate homotopy classes

$$[f] * [g] = [f * g] \quad [\text{independence on representatives see ex. exercises}]$$

Concatenation of homotopy classes has a group structure [group is set with binary operation $*$ with]

associativity $[f] * ([g] * [h]) = ([f] * [g]) * [h]$ exercises

$$f, g, h \in \mathcal{L} \quad [f * g * h] = [g * h * f] = [h * f * g] \quad \text{order does not matter}$$

$$[\text{point contractible loop}] = [\text{point contractible loop}]$$

$$[f]^{-1} = [f^{-1}] \quad [f] * [f^{-1}] = [\text{point contractible}]$$

comment
loops themselves do not form a group (no inverse)

π_1 of X If X is path connected (for every $x_0, x_1 \in X$ a continuous path ℓ_{x_0, x_1} in X), then

the loop $\ell_{x_0, x_1} * \ell_{x_1, x_0}^{-1}$ constructs isomorphism between the $\pi_1(x_0)$ & $\pi_1(x_1)$, by mapping the loops underlying both to each other

$$f: \text{loop at } x_0 \quad \ell_{x_0, x_1}^{-1} * f * \ell_{x_0, x_1} \text{ loop at } x_1$$

$\Rightarrow$ The fundamental group is a property of the path connected space itself

$$\pi_1(X).$$

$$\pi_1(x_0) \xrightarrow{\text{not cont.}} \pi_1(x_1) \xrightarrow{\text{cont. path } f: [0, 1] \rightarrow X \text{ cont.}} \pi_1(x_1)$$

not path. conn. path connected

II Topological classification of gapped, noninteracting, electronic phases

2006 Nobel prize

chiral (1980) Quantum Hall effect von Klitzing 1985

fractional qu. Hall effect Laughlin, Störmer, Tsui 1988

(2006) qu. spin Hall effect (Molenkamp, Zhang, Kane)

general & spin physics Kostelich, Thouless, Haldane

Majorana mode (Kitaev?)

Majorana bound state = nonabelian anyon