Lecture II — Topological insulators/supercond & fundamental symmetries

Lecture III — $\mathcal{H}(x,t)$ is cont. → π_1 group fundamental group π_1 — $\mathcal{H}(x,0) = f(x)$, $\mathcal{H}(x,1) = \mathbb{1}(x)$

§ 2.1 Why topl insulators are topological?

General Hamiltonians: single particle, relativistic, different symbols

$\mathcal{H} = \sum_{i,j\in L} A_{ij} c_i^\dagger c_j + \sum_{i,j\in L} B_{ij} c_i^\dagger c_j^\dagger + B_{ij}^\dagger c_i c_j$ + const.

c, linear operators on Hilbert space (Fock space)

$\{c_i^\dagger, c_j\} = c_i^\dagger c_j + c_j c_i^\dagger = \delta_{ij}$ — commut relations for fermions

$\{c_i, c_j\} = 0$

Since $\mathcal{H}^\dagger = \mathcal{H}$, we obtain

$\sum_{ij} A_{ij} c_i^\dagger c_j + \sum_{ij} A_{ij}^* c_i c_j^\dagger + \sum_{ij} B_{ij}^\dagger c_i^\dagger c_j = \sum_{ij} A_{ji}^* c_i^\dagger c_j + \sum_{ij} B_{ij} c_i c_j^\dagger$ + const

$\sum_{ij} B_{ij}^\dagger c_i^\dagger c_j + \sum_{ij} B_{ij} c_i c_j^\dagger = \sum_{ij} B_{ij} c_i^\dagger c_j^\dagger + \sum_{ij} B_{ij}^\dagger c_i c_j^\dagger$ — in odd cont in B

$\Rightarrow B_{ij} = -B_{ji}$

$c_i^\dagger c_j^\dagger$ — annihilation terms, represent suppressed. → Tikhonov limit in BCS theory

$\mathcal{H}$ is "diagonalizable", meaning that there are fermions $\alpha_j = \sum_{i\in L} \psi_{ji} c_i - \beta_{ji} c_i^\dagger$ such that

$\mathcal{H} = \sum_{j\in I} \varepsilon_j \alpha_j^\dagger \alpha_j$ + const | $\alpha_j^\dagger$ "Bogoliubons"

α_j — number operators

The transformation to achieve this is the Bogoliubov transformation

Insulator/gap: states of non-zero energy

Metal: Bosons $[b_q, b_q^\dagger] = \delta_{qq}$

Metal-vacuum for top phases

Vacuum

Metal

Boundary

Trivial phase [holds all states energy separated from 0 by gap]

Topological insulator/supercond phase

An insulator/supercond (gapped single particle excitations) is topological (or non-trivial) if there is no continuous transformation (of the parameters) that connects without a topological phase transition

Additional symmetries enrich the top phase diagram

$\mathcal{H}: \mathbb{R}^d \to \mathbb{C}^{N\times N}$

$\mathcal{H}(k) = -\mathcal{H}(-k)$

additional linear symmetry at $j=0$

§ 2.2 The bulk-boundary correspondence

Bulk

Boundary

Interface region

$\chi_I \mid \mathcal{H}(i\partial_x + \chi_I \cdot 0)\chi_I = \varepsilon\chi_I$

Bulk Class

Insulator for bulk x

Insulator for bulk x

Bulk boundary correspondence

Two top. insulators/sc. with different topological phase cannot together (in real space) connect to a metallic boundary

Δ_x

1 DOF - strong magnetic field

Su-Schrieffer-Heeger model (1D) excitation

$\mathcal{H} = \sum_j t c_j^\dagger c_{j+1} + h.c.$

Kitaev chain (1D)

$\mathcal{H} = \sum_j -\mu c_j^\dagger c_j - t c_j^\dagger c_{j+1} + \Delta c_j c_{j+1} + h.c.$

$t = \Delta$ — two topologically distinct phases

§ 2.3 The twofold way — classification of the matter

Gaplode

Wigner theorem (ASSA)

Symmetries in a QM system are represented by:

1) unitary operators U or $\mathcal{U}$ if $|\langle\psi_1|\psi_2\rangle|^2 = |\langle U\psi_1|U\psi_2\rangle|^2$

2) anti-unitary operators $\mathcal{U}$ defined by anti-linearity $\mathcal{U}(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha^*\mathcal{U}|\psi_1\rangle + \beta^*\mathcal{U}|\psi_2\rangle$ & $|\langle\mathcal{U}\psi_2|\mathcal{U}\psi_1\rangle| = |\langle\psi_2|\psi_1\rangle|$

$\Rightarrow |\langle\psi_1|\psi_2\rangle|^2 = |\langle\psi_1|\mathcal{U}^\dagger\mathcal{U}|\psi_2\rangle|^2 = |\langle\psi_1|\mathcal{U}^\dagger|\mathcal{U}\psi_2\rangle|^2$ → Wikiway 2002

complex conjugation operator

We have $K: \mathbb{C} \to \mathbb{C}$ with $z \mapsto z^*$

Define $K: \mathcal{H} \to \mathcal{H}$ by choosing a w.r.t. Hilbert space basis $\{|\psi_j\rangle\}_{j\in I}$ & $K|\psi\rangle = \sum_{j\in I} \psi_j^* |\psi_j\rangle$ $\Leftrightarrow |\psi\rangle = \sum_{j\in I} \psi_j |\psi_j\rangle$

Attention (Strongly) dependent on basis.

Imagine different basis $|\psi_j'\rangle = |\psi_j\rangle, |\psi_j\rangle = |\psi_j\rangle$

$|\psi\rangle = \sum_j \psi_j |\psi_j\rangle = \sum_j (-\psi_j)|\psi_j\rangle$

$K\sum_j \psi_j |\psi_j\rangle = \sum_j \psi_j^* |\psi_j\rangle$

but $K\sum_j (-\psi_j)|\psi_j\rangle = \sum_j (-\psi_j)^* |\psi_j\rangle = \sum_j -\psi_j^* |\psi_j\rangle$

Attention Typically K is used without specification of "canonical bases" exists.

Representation of anti-unitary operators

$\mathcal{U} = U \cdot K$, where K is compl. conjugation & U is unitary.

proof: Given $\mathcal{U}$, consider $\mathcal{U}K$ is unitary (group axiom) then

$\mathcal{U} = \overline{(\mathcal{U}K)}\, K$ ☐

$\langle\psi_1|\psi_2\rangle = \sum_j \psi_j^* \psi_j$

$\langle\mathcal{U}K\psi_1 \mid \mathcal{U}K|\psi_2\rangle$

$= \overline{\langle\mathcal{U}K\psi_1|}\,\mathcal{U}K|\psi_2\rangle$

$= \overline{\sum_j \psi_j^* \mathcal{U}K|\psi_j\rangle}\,\mathcal{U}K|\psi_2\rangle$

$= \sum_j \psi_j \overline{\mathcal{U}K|\psi_j\rangle}\,\mathcal{U}K|\psi_2\rangle$

$= \sum_j \psi_j \mathcal{U}|\psi_j\rangle\,\mathcal{U}K|\psi_2\rangle$

$= \sum_j \psi_j |\psi_j\rangle = \langle\psi_1|\psi_2\rangle$ → unitarity ☐

§ 2.2.2 Scheme for classification

Given a diff'ble manifold $\mathcal{M}$ & its symmetries by unitary & anti-unitary operators, the twofold way follows the following steps. Let $\mathcal{H}$ be a representation of $\mathcal{M}$

1) Reduce all unitary symmetries

unitary symmetry U $[\mathcal{H}, U] = 0$

→ $\mathcal{H} = \begin{pmatrix} \mathcal{H}_1 & & & \\ & \mathcal{H}_2 & & \\ & & \ddots & \\ & & & \mathcal{H}_n \end{pmatrix}$ — general quantum number

where U acts trivially (like mult. of $\mathbb{1}$ with coefficient) on each block

$\mathcal{H}_1: \mathcal{U}_1 \mathcal{H}_1 = \mathcal{H}_1 \mathcal{U}_1$ — restricted