Section VI cont

classification of top phases

Def Σ · unitary & antiunitary symmetry
· family of Hamiltonians $\mathcal{H}$ with
same Σ and unitary symmetries

I Ideal diagonalize Hamiltonians

$\mathcal{H} = \begin{pmatrix} \mathcal{H}^1 & 0 & 0 \\ 0 & \mathcal{H}^2 & 0 \\ 0 & 0 & \mathcal{H}^3 \end{pmatrix}$

1) each unitary symmetry prop to 1 on individual blocks

2) classify each ideal $\mathcal{H}^i$ individually

All. Find
T a) the several symmetry T with $[\mathcal{H},T]=0$ & antiunit
C b) particle hole symmetry P with $\{\mathcal{H},P\}=0$ & unitary
S c) chiral symmetry C with $\{\mathcal{H},C\}=0$ & unitary

(alternative Labels) ... all acting on single particle space

Canonical examples

time reversal symmetry: $T \propto \mathbb{C}^1 \hat{T}^1 \cdot \mathcal{K}^1 \mathbb{C}^1$ | $T \cdot \mathcal{K}$
for spinless systems $[\mathcal{H},T]=0$ for all k: $U \cdot \mathbb{C}^1 \mathcal{H}^1 \mathbb{C}^{1\dagger} U^\dagger = \mathcal{H}^1(-k)$, $TUT^{-1}\mathbb{C}^1 \cdot \mathcal{K}^1 \mathbb{C}^{1\dagger} U^\dagger \mathbb{C}^1 \mathcal{K}^1 \mathbb{C}^{1\dagger} = \mathcal{H}^1(+k)$
$\mathcal{H}^1 = T\mathcal{H}\cdot T^{-1} = K$ ... and
... for spinful systems: $T\mathbb{C}_1\hat{T}^{1*}\cdot\mathcal{K}^1\mathbb{C}_1$, $T\mathbb{C}_1\hat{T}^{1*}\cdot\mathcal{K}^1\mathbb{C}_1$ | $T_{\text{spinful}} = T_{\text{spinless}} \otimes \mathbb{1}_{2\times 2} \cdot K$

Particle hole symmetry

$P \propto \mathbb{C}\hat{P}^{1*}\cdot\mathcal{K}^1\mathbb{C}$.

Properties of T, P, C

a) T, P, C are unique up to scalar]!!

proof for T, all others analogously

Let $T \& T'$ be time reversal symmetries of $\mathcal{H}, \mathcal{H}^1$
Then TT' is unitary because
if $T = A\mathcal{K}\hat{T}\cdot A'\mathcal{K}$, then

$TT' = A\mathcal{K}^1\mathcal{A}'^{\dagger}$ with A, A' unitary

$(TT')(TT')^* = A\underbrace{\mathcal{K}^1\mathcal{A}'^{\dagger}\mathcal{K}^1}_{(\mathcal{K}^1\mathcal{K})^2=\mathbb{1}}A^{\dagger} = AA'^{\dagger}\cdot\mathbb{1}$ $\quad \begin{bmatrix} [T,\mathcal{H}]=0 \\ T[T',\mathcal{H}]T'^{-1}=[T,\mathcal{H}]=0 \end{bmatrix}$

All unitary symmetries on $\mathcal{H}^1$ are $\propto \mathbb{1}$
$\Rightarrow TT' = e^{i\varphi}\mathbb{1} \Rightarrow T = e^{i\varphi}(T')^{-1}(*)$

In particular $T^2=\pm\mathbb{1}$, see (E) below
$\Rightarrow T^{-1}e^{i\varphi}T = e^{i\varphi}T \Rightarrow$ TRS only differ by scalar factor

b) $T^2=\pm\mathbb{1}$, $P^2=\mathbb{1}$, $C^2=\mathbb{1}$ (w.l.o.g. can set $=\mathbb{1}$)

See (a) & set $T=T'$
$T = e^{i\varphi}TT' \Rightarrow T^2 = e^{i\varphi}$ with $T=A\mathcal{K}$
$\Rightarrow A\underbrace{A^*}_{\mathbb{1}}e^{i\varphi}\mathbb{1}$. Take trace

$\operatorname{Tr}\frac{1}{2}AA^*\xi = e^{i\varphi}n$ (E)

but also (for other complex conjugation)
$\operatorname{tr}\frac{1}{2}A^*A\xi = e^{-i\varphi}n = \operatorname{tr}\frac{1}{2}A\xi\cdot e^{-i\varphi}n$ (E')
should (E)=(E') $\Rightarrow$
$\lambda = e^{2i\varphi} \Rightarrow \varphi \in \pi\mathbb{Z}$

$\Rightarrow T^2 = e^{i\varphi} \in \{-\mathbb{1}, \mathbb{1}\}$

$\Rightarrow T^2 = \pm\mathbb{1}$ (*)

$[AB,C] = ABC - CAB = \underbrace{\{A,C\}}_{=0}B - A\underbrace{[B,C]}_{=0}$
$= A[B,C] = [A,C]B$ (ex 2)

$[AB,C] = \underbrace{\{A,C\}}_{=0}B - A\{B,C\}$ (ex 2)

...

proof $C^2=\mathbb{1}$ (ex) can be chosen
consider $[C^2,\mathcal{H}] \overset{C}{=} C\underbrace{\{C,\mathcal{H}\}}_{=0}\xi - \xi\underbrace{\{\mathcal{H},C\}C}_{=0} = 0$

$\Rightarrow C^2 = e^{i\varphi}\mathbb{1} \Rightarrow$ choose $\tilde{C} = e^{i\varphi/2}C$
Then $\tilde{C}^2 = e^{i\varphi}C^2 = e^{i(\varphi+\varphi)} = \mathbb{1}$.

Properties of $\tilde{C}$
Since $\tilde{C}^2=\mathbb{1} \Rightarrow \tilde{C}$ eigenvalues are $\pm\mathbb{1}$
Def $|x\rangle$ be eigenbasis of $\tilde{C}$, then express $\mathcal{H}$ in that basis & consider

$\langle x|\mathcal{H}|\tilde{x}\rangle$ with $C|\tilde{x}\rangle = |\tilde{x}\rangle$ &
$C|x\rangle = -|x\rangle$ &

Then $\langle x|\xi\underbrace{\tilde{C}\mathcal{H}C}_{-\mathcal{H}}\xi|\tilde{x}\rangle$ $\Big|$ because $\{\tilde{C},\mathcal{H}\}=\{C,\mathcal{H}\}=0$ & $\tilde{C}=C$
$= -\xi^2\langle x|\mathcal{H}|\tilde{x}\rangle = -\langle x|\mathcal{H}|\tilde{x}\rangle \Rightarrow \langle x|\mathcal{H}|\tilde{x}\rangle = 0$

$\Rightarrow \mathcal{H}_{x\tilde{x}} = \langle x|\mathcal{H}|\tilde{x}\rangle \cdot \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}$ $\begin{matrix} \leftarrow x \\ \leftarrow \tilde{x} \end{matrix}$ eigen ± 1
$\Rightarrow$ chiral symmetries
naturally not chiral subsectors
("opposite to unitary symmetries which related/duplicating $\mathcal{H}$")

c) Kramer's degeneracy

Given a TRS T with $T^2=-\mathbb{1}$
Consider an eigenstate $|\psi\rangle$ with
$\mathcal{H}|\psi\rangle = \varepsilon|\psi\rangle$. Then

$\mathcal{H}T|\psi\rangle = T\mathcal{H}|\psi\rangle = T\varepsilon|\psi\rangle = \varepsilon T|\psi\rangle$.
$[\mathcal{H},T]=0 \Rightarrow \mathcal{H}T=T\mathcal{H}$
$\Rightarrow T|\psi\rangle$ is another state with same energy (if $\langle\psi|T|\psi\rangle=0 \Rightarrow$ see exercises)

d) particle hole symmetry spectrum

Take a PHS P & consider eigenvector $\mathcal{H}|\psi\rangle = \varepsilon|\psi\rangle$
Then
$\mathcal{H}P|\psi\rangle = -P\mathcal{H}|\psi\rangle = -P\varepsilon|\psi\rangle = -\varepsilon P|\psi\rangle$
$\{\mathcal{H},P\}=0 \Rightarrow P\mathcal{H}P=-\mathcal{H}$

Total possibilities [$T^2=\pm\mathbb{1}$, $P^2=\pm\mathbb{1}$, $C^2=\mathbb{1}$ & $TP=\pm C$; $CP=\pm T$; $C=TP$] (by exercises)

symmetry class (Cartan)	T^2	P^2	C^2
A (unitary)	0	0	1
AIII	0	0	1
AI	1	0	1
BDI	1	1	1
D	0	1	1
DIII	-1	1	1
AII	-1	0	1
C	-1	-1	1
CI	1	-1	1
	0	-1	1

$T^2=0$ means no symmetry of $T^2=1$ case basically $T^2=+1$

$\Rightarrow$ tenfold way

X. Vbdl subspace fibre bundles

Consider a $\mathcal{H}$.Vbdl space $\mathcal{H}.\text{Vb}$ & a parameter manifold $\mathcal{B}$ (locally trivial Hausdorff), and a continuous map $f: \mathcal{B} \to \mathcal{H}(\mathcal{H}.\text{Vb} \to \mathcal{H}.\mathcal{B})$ from $\mathcal{B}$ to the bounded operators acting on $\mathcal{H}.\text{Vb}$ with a minimal eigenvalue. This means operators have exactly eigenvalues & eigenvectors in step on parameters in $\mathcal{B}$

Let $\mathcal{F}(q)$ for $q \in \mathcal{B}$ be the subspace of $\mathcal{H}.\text{Vb}$ spanned by the n eigenvectors with the lowest energy of f_q.

Then $E = \{(q,|\psi\rangle) \mid q \in \mathcal{B}, |\psi\rangle \in \mathcal{F}(q)\}$
is a subset of $\mathcal{B} \times \mathcal{H}.\text{Vb}$ (gets a product space topology) & $E \overset{\pi}{\to} \mathcal{B}$ with $\pi: E \to \mathcal{B}$, $(q,|\psi\rangle) \mapsto q$ is a fibre bundle.

reduction