

Lecture 3 Majorana invariant & adiabatic state manipulation (connections to quantum computing)

Lecture 8. Example for top. SC: Kitaev chain
Kitaev 2001.

$$\mu c_j^\dagger c_j + \sum_j \Delta c_j^\dagger c_{j+1} + w c_j^\dagger c_{j+1}$$

type P 1d top. supercond.
no TRS, $P^2 = 1$ (no C)

Topological invariant of Kitaev chain

$$\mathcal{K} = \sum_{i,j} \tilde{f}_i^T A_{i,j} \tilde{f}_j = \sum_{i,j} \frac{1}{2} \text{ with } A^T = -A$$

A is $2N \times 2N$ matrix

$$\det(A) \neq 0, Pf(A), Pf^2(A) = \det(A) \rightarrow \text{look } \text{sgn}(Pf(A)).$$

calculate Pfaffian of A

$$A_{i,j}^{block} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & -(\Delta + w) \\ (\Delta + w) & 0 \end{pmatrix}$$

$$B(q) = \sum_{\delta} e^{iq\delta} A(\delta_j) \quad q \in \{1, \dots, N\} \frac{2\pi}{N}$$

$$\Rightarrow Pf(A) = \prod_{q \in \text{BZ}} Pf(B(q)) \prod_{q \notin \text{BZ}} \det(B(q))$$

We see that

$$B(q) = \begin{pmatrix} 0 & \mu + \Delta(e^{iq} - e^{-iq}) + w(e^{iq} - e^{-iq}) \\ -\mu + \Delta(e^{iq} - e^{-iq}) + w(e^{iq} - e^{-iq}) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mu + 2i\Delta \sin(q) - 2w \cos(q) \\ -\mu + 2i\Delta \sin(q) + 2w \cos(q) & 0 \end{pmatrix}$$

$$\Rightarrow \det(B(q)) = -(-4\Delta^2 \sin^2(q) - (\mu - 2w \cos(q))^2)$$

$$= (2\Delta \sin(q))^2 + (\mu - 2w \cos(q))^2 \geq 0$$

[comment: This is dispersion relation/spectrum (square)]

$$\Delta > 0 \quad \text{only vanish if } \mu = 2w$$

$$\Rightarrow \text{sgn } Pf(A) = \text{sgn} \left(Pf \begin{pmatrix} 0 & \mu - 2w \\ 2w & 0 \end{pmatrix} Pf \begin{pmatrix} 0 & \mu + 2w \\ \mu + 2w & 0 \end{pmatrix} \right)$$

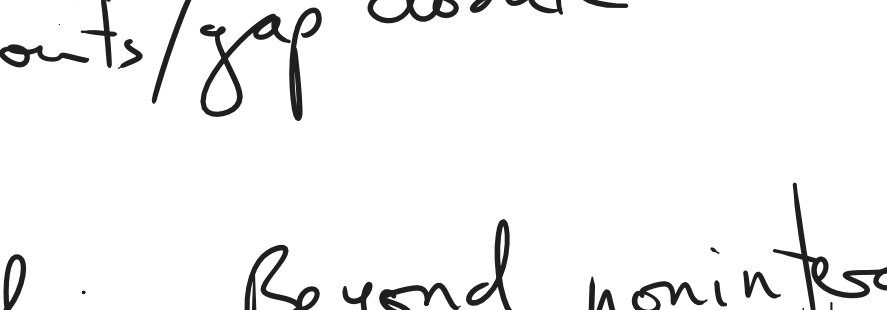
$$= \text{sgn}(\mu - 2w)(\mu + 2w) = \text{sgn}(\mu^2 - 4w^2)$$

$$= \text{sgn}(|\mu| - 2w)$$

$\Rightarrow$ top. invariant

$$Z = \begin{cases} 1 & |\mu| > 2w \\ -1 & |\mu| < 2w \end{cases}$$

triv. non triv.



Remark: Beyond noninteracting gapped top. electronic matter

• top. semi-metals (Dirac/Weyl metals)

• Non-hermitian topological effects (pump systems/classical mechanical/optical systems)

• Interacting top. systems $\rightarrow$ fractional quantum Hall effect.

• Kondo's insulators $\rightarrow$ quantum Hall effect.

• basic topological phases

Motivation: Weyl semi-metals

$$\mathcal{H} = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger \left(d_0(\mathbf{k}) + d_1(\mathbf{k}) \sigma_x + d_2(\mathbf{k}) \sigma_y + d_3(\mathbf{k}) \sigma_z \right) \psi_{\mathbf{k}}$$

$$\mathcal{E} = d_0(\mathbf{k}) \pm \sqrt{d_1^2(\mathbf{k}) + d_2^2(\mathbf{k}) + d_3^2(\mathbf{k})}$$

band crossing depends on 3 real parameters

$\Rightarrow$ 3D generic scenario: lowest bands touch same, singular point, in the BZ.

$$\mathcal{E} = d_0(\mathbf{k}) \pm \sqrt{d_x^2 k_x^2 + d_y^2 k_y^2 + d_z^2 k_z^2} \quad \text{by linearly expanding } d(\mathbf{k}).$$

top. charge -1

Weyl node (no degeneracy)

Dirac node (degenerate twice)

like top. phase transition in dim d-1.

cut out surface in k-space that incl. Weyl cone $\Rightarrow$ d-1 gapped system

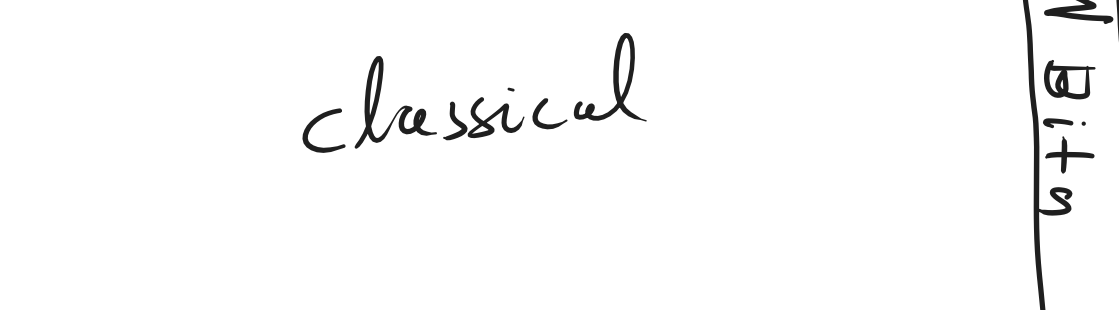
$\Rightarrow$ classify according to topological index is "topological charge of Weyl node".

III Adiabatic state manipulation & topological & adiabatic quantum computing

adiabatic: $\alpha \Delta \beta \alpha \Delta \beta \alpha \Delta \beta$

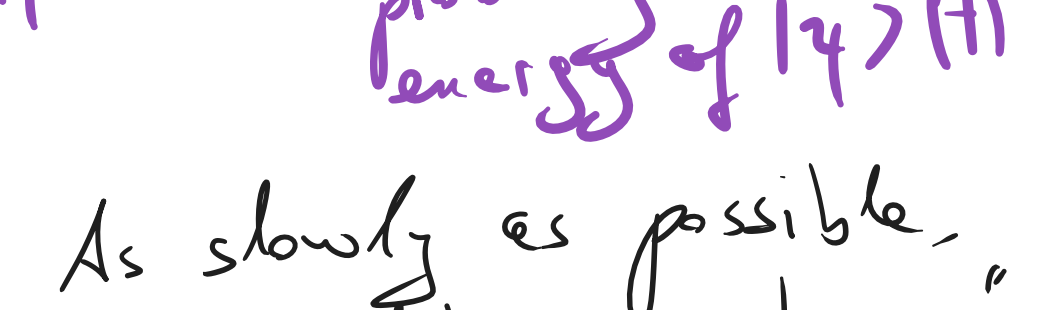
thermodynamics not through

change of state without heat transfer



As fast as possible "heat cannot pass"

quantum mechanics time-dependent Hamiltonian $\mathcal{H}(t)$



As slowly as possible, "probability cannot pass".

Computing

classical

$$\{0,1\}^N$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$$

$$\xrightarrow{f}$$

$$\{0,1\}^N \rightarrow \{0,1\}^M$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$$

• prepare input

• run calculation (apply f)

• read out result

smallest unit of data: Bit $\{0,1\}$.

calculation: Decompose f into gate

$$\rightarrow \frac{01}{10} \text{ \& \> } \frac{010}{0101} \text{ \& \> } \frac{1101}{11011}$$

complete set of gates: Can construct all f.

quantum computer

dim. Hilb. space 2^N

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \xrightarrow{f: \mathcal{H}_1 \rightarrow \mathcal{H}_2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}$$

unitary

1) prepare input state

2) conduct f (time evol)

3) read out (measurement)

qubit: 2-dim complex Hilb. space.

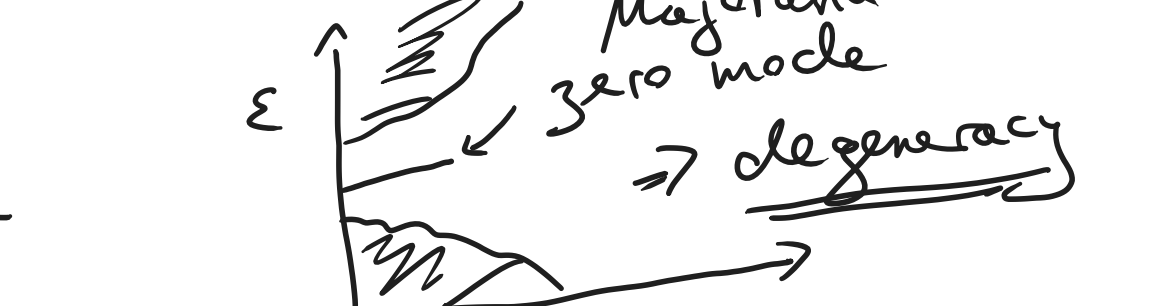
convenient tool: Pauli matrices

group $SU(2)$ generated by Pauli matrices

$U = e^{i\frac{\theta}{2} \sigma}$ $SU(2)$ is a double cover of $SO(3)$ (the rotations)

$\hookrightarrow$ Pauli gates $SO(3)$ (the rotations)

$\Rightarrow$ Bloch sphere



quantum gates: Building blocks of unitary time evolution

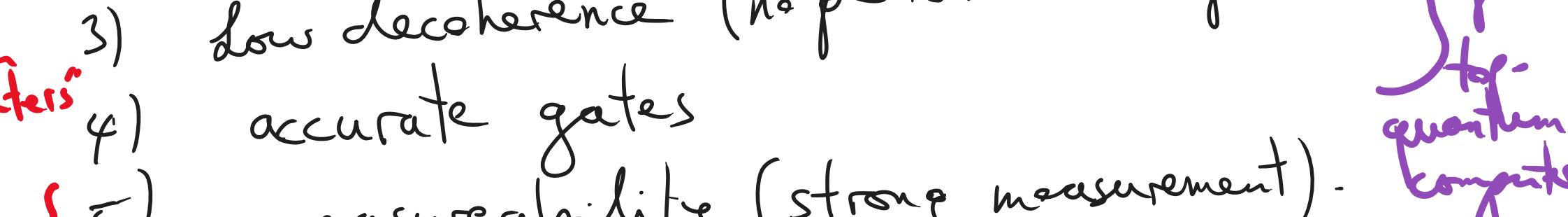
universal quantum gates: Arbitrary close approximation of any unitary transform possible

e.g.: Pauli gates (1 qubit gates), CNOT gate (2 qubit gate)

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

How to use adiabatic manipulation for quantum computing

e.g. Kitaev chain



unitary time evolution $\hat{=}$ quantum gate

• gate is protected by topology.

Di Vincenzo criteria (1998)

- 1) identification well-defined qubits
- 2) reliable state preparation
- 3) low decoherence (no perturbations of state)
- 4) accurate gates
- 5) measurability (strong measurement).

"normal quantum computers"

Google Willow 105 qubits

IBM Heron 133 qubits

play for top. quantum computers